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POPULATION MODELS WITH NONLINEAR DIFFUSION OF KOLMOGOROV-FISHER TYPE

D. K. MUHAMEDIYEVA

Junior Researcher, Development of Software Products, TUIT, Tashkent, Uzbekistan

ABSTRACT

Consider the model of two competing species with non-linear diffusion and three types of functional dependencies. The first type of dependence corresponds to Malthusian demographic processes, the second - ferhulst (logistic population), and the third - the population of "Allee". The common element to this kind of description is the presence of a linear source, and in the descriptions of the populations of Ferhulst and Allee type are also nonlinear sinks.

Offered suitable initial approximations for quickly convergence iterative process. Numerical experiments are carried out with visualization for different values of parameters of the system of biological population of Kolmogorov-Fisher type.

KEYWORDS: Biological Population, Nonlinear System of Partial Differential Equations, Initial Approximation, Numerical, Iterative Process, Self-Similar Solutions

INTRODUCTION

Blow-up regimes in spatial localization in open dissipative systems described by models with nonlinear diffusion [1-2]. Some cross-diffusion models discussed in several papers in physical systems (plasma physics) [2,3], in chemical systems (dynamics of electrolyte solutions [4]), biological systems (cross-diffusive transport [5], the dynamics of population systems [6, 7-8], in ecology (return dynamics of forest structure [9,10]), in seismology model Burridj-Knopoff describing the interaction of tectonic plates [11-12]. In the last ten years in the study of the growth and development of tumors actively used mathematical models with cross-diffusion [13-14] (as well as the system of reaction-diffusion-advection [14]).

Let us explain what we mean by the term "cross-diffusion" (or cross-diffusion). Consider the following system of two equations in one-dimensional case:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= f(u_1,u_2) + D_1\frac{\partial^2 u_1}{\partial x^2} + h_1\frac{\partial}{\partial x}\left(Q_1(u_1,u_2)\frac{\partial u_2}{\partial x}\right) \\ \frac{\partial v}{\partial t} &= g(u_1,u_2) + D_2\frac{\partial^2 u_2}{\partial x^2} + h_2\frac{\partial}{\partial x}\left(Q_2(u_1,u_2)\frac{\partial u_1}{\partial x}\right) \end{aligned} \quad (1)$$

At $h_1 = h_2 = 0$ mathematical model (1) is a system of reaction-diffusion with the diffusion coefficients $D_1 \geq 0,\ D_2 \geq 0$ (at least one $D_i \neq 0$). In the case where at least one of the coefficients $h_i \neq 0$ (_sign can be any), system (1) is a cross-diffusion.

Linear cross-diffusion corresponds to $Q_i(u_1,u_2)=const$ for i=1,2; for nonlinear cross-diffusion- $Q_i(u_1,u_2)\neq const$ at least one i.

Cross-diffusion means a spatial displacement of an object described by one of the variables takes place by diffusion of another object, the other variable described. At the population level, the simplest example of a parasite (the first item), located inside the «host» (the second object moves through the diffusion of the owner). The term «self-diffusion» (diffuse, direct diffusion, ordinary diffusion) involves the movement of individuals at the expense of the diffusion flow from areas of high concentration, particularly in the area of low concentration. The term "cross-diffusion" refers to the movement / flow of individuals of one species / substances due to the presence of the gradient of other individuals / agents. The magnitude of cross-diffusion coefficient can be positive, negative or zero. Positive cross-diffusion coefficient indicates that movement of individuals is in a direction other species of low concentration occurs in the direction of high concentrations of other species of animals / substances. In nature, systems with cross diffusion are quite common and play an important role, especially in biophysical and biomedical systems.

Equation (1) is a generalization of the simplest diffusion model for the logistic model of population growth [16-20] in Malthusian type ($f_1(u_1,u_2)=u_1$, $f_1(u_1,u_2)=u_2$, $f_2(u_1,u_2)=u_1$, $f_2(u_1,u_2)=u_2$), in Ferhulst type ($f_1(u_1,u_2)=u_1(1-u_2)$, $f_1(u_1,u_2)=u_2(1-u_1)$, $f_2(u_1,u_2)=u_1(1-u_2)$, $f_2(u_1,u_2)=u_2(1-u_1)$), and Allee type ($f_1(u_1,u_2)=u_1(1-u_2{}^{\beta_1})$, $f_1(u_1,u_2)=u_2(1-u_1{}^{\beta_2})$, $f_2(u_1,u_2)=u_1(1-u_2{}^{\beta_1})$, $f_2(u_1,u_2)=u_2(1-u_1{}^{\beta_2})$, $\beta_1>1,\ \beta_2>1$) for the case of double nonlinear diffusion.

In case, when $\beta_1\geq 1,\ \beta_2\geq 1$, it can be viewed also as the equation of nonlinear filtration, thermal conductivity while the impact source and absorption capacity which are respectively $u_1,-u_2{}^{\beta_1}$, $u_2,-u_1{}^{\beta_2}$

Consider a spatial analogue of Volterra-Lotka competition system with a nonlinear power dependence of the diffusion coefficient of density of population. In the case of the simplest volterra competitive interactions between populations can be constructed numerically, and in some cases analytically heterogeneous in space solutions [21].

STATEMENT OF THE TASK

Consider in the domain Q={(t,x): 0< t < ∞, $x\in R^2$} parabolic system of two quasilinear equations of reaction-diffusion task of biological populations of Kolmogorov-Fisher type.

$$\begin{cases} \dfrac{\partial u_1}{\partial t}=\dfrac{\partial}{\partial x}\left(D_1 u_2^{\sigma_1}\dfrac{\partial u_2}{\partial x}\right)+k_1(t)u_1\left(1-u_2^{\beta_1}\right) \\ \dfrac{\partial u_2}{\partial t}=\dfrac{\partial}{\partial x}\left(D_2 u_1^{\sigma_2}\dfrac{\partial u_1}{\partial x}\right)+k_2(t)u_2\left(1-u_1^{\beta_2}\right) \end{cases} \tag{2}$$

$$u_1\big|_{t=0}=u_{10}(x),\quad u_2\big|_{t=0}=u_{20}(x),$$

that describes the process of biological populations in a nonlinear two-component environment, the diffusion coefficient is equal to $D_1 u_2^{\sigma_1}$ and $D_2 u_1^{\sigma_2}$, $\sigma_1, \sigma_2, \beta_1, \beta_2$ - positive real numbers, $u_1 = u_1(t,x) \ge 0$, $u_2 = u_2(t,x) \ge 0$ - required solutions.

The Cauchy problem and boundary problems for the system (1) in the univariate and multivariate cases investigated by many authors [17-23].

The aim of this work is to study the qualitative properties of solutions of the problem (2) on the basis of the self-analysis and numerical solutions using the methods of modern computer technology, research methods of linearization to the convergence of the iterative process with further visualization. Found estimates for solutions and the resulting free boundary, that gives the chance to choose the appropriate initial approximation [17] for each of the values of numerical parameters.

Let's build system of equations (1)-(2) - more simple system of equations for research.

CONSTRUCTION SELF-SIMILAR SYSTEM OF EQUATIONS

Let's construct a system of equations by the method of nonlinear splitting [3].

Replacing in (2)

$$u_1(t,x) = e^{k_1 t} v_1(t,x),$$
$$u_2(t,x) = e^{k_2 t} v_2(t,x)$$

lead (2) to the form:

$$\begin{cases} \dfrac{\partial v_1}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_1 v_2^{\sigma_1} \dfrac{\partial v_2}{\partial x} \right) - k_1(t) e^{(\beta_1 k_2 - \sigma_1 k_2 - k_2 + k_1)t} v_1 v_2^{\beta_1} \\ \dfrac{\partial v_2}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_2 v_1^{\sigma_2} \dfrac{\partial v_1}{\partial x} \right) - k_2(t) e^{(\beta_2 k_1 - \sigma_2 k_1 - k_1 + k_2)t} v_1^{\beta_2} v_2 \end{cases} \quad (3)$$

$v_1\big|_{t=0} = v_{10}(x)$, $v_2\big|_{t=0} = v_{20}(x)$,

By choosing $\sigma_1 k_2 + k_2 - k_1 = \sigma_2 k_1 + k_1 - k_2$, we obtain the following system of equations,

$$\begin{cases} \dfrac{\partial v_1}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_1 v_2^{\sigma_1} \dfrac{\partial v_1}{\partial x} \right) - a_1(t) \tau^{b_1} v_1 v_2^{\beta_1} \\ \dfrac{\partial v_2}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_2 v_1^{\sigma_2} \dfrac{\partial v_2}{\partial x} \right) - a_2(t) \tau^{b_2} v_1^{\beta_2} v_2 \end{cases} \quad (4)$$

where $a_1 = k_1(\sigma_1 k_2 + k_2 - k_1)^{b_1}$, $b_1 = \dfrac{\beta_1 k_2 - \sigma_1 k_2 - k_2 + k_1}{\sigma_1 k_2 + k_2 - k_1}$,

$$a_2 = k_2(\sigma_2 k_1 + k_1 - k_2)^{b_2}, \qquad b_2 = \frac{\beta_2 k1_2 - \sigma_2 k_1 - k_1 + k_2}{\sigma_2 k_1 + k_1 - k_2}.$$

If $b_i = 0$, $i = 1,2$, the system has the form:

$$\begin{cases} \dfrac{\partial v_1}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_1 v_2^{\sigma_1} \dfrac{\partial v_1}{\partial x}\right) - a_1(t) v_1 v_2^{\beta_1} \\ \dfrac{\partial v_2}{\partial \tau} = \dfrac{\partial}{\partial x}\left(D_2 v_1^{\sigma_2} \dfrac{\partial v_2}{\partial x}\right) - a_2(t) v_1^{\beta_2} v_2 \end{cases}$$

The Cauchy problem for system (4) in case, when $b_1 = b_2 = 0$ studied in the works [18, 21] and proved the existence of a wave of global solutions and blow-up solutions.

Below we will describe one way of getting self-similar system for the system of equations (4). It consists in the following. Find the first solution of a system of ordinary differential equations

$$\begin{cases} \dfrac{d\bar{v}_1}{d\tau} = -a_1 \bar{v}_1 \bar{v}_2^{\beta_1} \\ \dfrac{d\bar{v}_2}{d\tau} = -a_2 v_1^{\beta_2} v_2 \end{cases}$$

in the form

$$\bar{v}_1(\tau) = c_1(\tau + T_0)^{-\gamma_1}, \quad \bar{v}_2(\tau) = c_2(\tau + T_0)^{-\gamma_2}, \quad T_0 > 0,$$

where

$$c_1 = 1, \quad \gamma_1 = \frac{1}{\beta_2}, \quad c_2 = 1, \quad \gamma_2 = \frac{1}{\beta_1}.$$

And then the solution of system (3)-(4) is sought in the form

$$v_1(t, x) = \bar{v}_1(t) w_1(\tau, x), \tag{5}$$

$$v_2(t, x) = \bar{v}_2(t) w_2(\tau, x),$$

and $\tau = \tau(t)$ selected as

$$\tau_1(\tau) = \int_0^\tau \bar{v}_1^{\sigma_1}(\tau) dt = \begin{cases} \dfrac{1}{1 - \gamma_1 \sigma_1}(T + \tau)^{1 - \gamma_1 \sigma_1}, & if \ \ 1 - \gamma_1 \sigma_1 \neq 0, \\ \ln(T + \tau), & if \ \ 1 - \gamma_1 \sigma_1 = 0. \end{cases}$$

Then for $w_i(\tau, x)$, $i = 1,2$ we get following system of equation

$$\begin{cases} \dfrac{\partial w_1}{\partial \tau} = \dfrac{\partial}{\partial x}(D_1 w_2^{\sigma_1} \dfrac{\partial w_2}{\partial x}) + \psi_1 (w_1 w_2^{\beta_1} - w_1) \\ \dfrac{\partial w_2}{\partial \tau} = \dfrac{\partial}{\partial x}(D_2 w_1^{\sigma_2} \dfrac{\partial w_1}{\partial x}) + \psi_2 (w_2 w_1^{\beta_2} - w_2) \end{cases}, \quad (6)$$

where

$$\psi_i = \begin{cases} \dfrac{\gamma_i}{(1-\gamma_1\sigma_1)\tau}, & if \ \ 1-\gamma_1\sigma_1 > 0, \\ \gamma_i c_1^{-\sigma_1}, & if \ \ 1-\gamma_1\sigma_1 = 0. \end{cases} \quad (7)$$

Representation of system (1) in the form (5) suggests that, when

$$\tau \to \infty, \ \psi_i \to 0 \ \text{and} \ \begin{cases} \dfrac{\partial w_1}{\partial \tau} = \dfrac{\partial}{\partial x}(D_1 w_2^{\sigma_1} \dfrac{\partial w_2}{\partial x}), \\ \dfrac{\partial w_2}{\partial \tau} = \dfrac{\partial}{\partial x}(D_2 w_1^{\sigma_2} \dfrac{\partial w_1}{\partial x}). \end{cases} \quad (8)$$

Therefore, the solution of system (2) with the conditions (5) tends to the solution of the system (8).

Let $\gamma_1\sigma_1 > 1$, $\gamma_1\sigma_1 = \gamma_2\sigma_2$, $\sigma_2(b_2+1) + \beta_2(b_1+1) = \sigma_1(b_1+1) + \beta_1(b_2+1)$, $c_i > 0$. In this case, by assuming in (6)

$$w_i(\tau(t), x) = y_i(\xi), \quad \xi = |x| / \tau_1^{1/2}, \ i = 1, 2,$$

and considering, that the equation $w_i(\tau, x)$ without the younger members always has a self-similar solution in the case of $1-\gamma_1\sigma_1 \neq 0$, we obtain a system

$$\begin{cases} \dfrac{d}{d\xi}(y_1^{\sigma_1} \dfrac{dy_1}{d\xi}) + \dfrac{\xi}{2}\dfrac{dy_1}{d\xi} + \mu_1(y_1 - y_1 \ y_2^{\beta_1}) = 0, \\ \dfrac{d}{d\xi}(y_2^{\sigma_2} \dfrac{dy_2}{d\xi}) + \dfrac{\xi}{2}\dfrac{dy_2}{d\xi} + \mu_2(y_2 - y_2 \ y_1^{\beta_2}) = 0, \end{cases}$$

where $\mu_i = \dfrac{\gamma_i}{1-\gamma_i\sigma_i}$.

The study of qualitative properties of the system (2) is allowed to perform the numerical experiment depending on the values included in the system of numerical parameters. For this purpose, as the initial approximation was used constructed asymptotic solutions.

The numerical solution of the problem for linearization of system (2) were used linearization methods of Newton and Picard. To build self-similar system of equations of biological population used the method of nonlinear splitting [18,21].

NUMERICAL EXPERIMENT

To solve the task (1)-(2) by numerical method we construct a uniform grid

$$\omega_h = \{x_i = ih, \ \ h > 0, \ \ i = 0,1,...,n, \ \ hn = l\},$$

and temporary grid

$$\omega_{h_1} = \{t_j = jh_1, \qquad h_1 > 0, \ \ j = 0,1,...,n, \ \ \tau m = T\}.$$

Replace the task (1)-(2) with implicit difference scheme and receive differential task with the error $O(h^2 + h_1)$.

As it is known, the main problem for the numerical solution of nonlinear problems is the suitable choice of the initial approximation and the method of linearization of system (2).

Let's consider the function:

$$v_{10}(t,x) = v_1(t) \cdot (a - \xi^2)_+^{\varphi_1},$$

$$v_{20}(t,x) = v_2(t) \cdot (a - \xi^2)_+^{\varphi_2},$$

where $v_1(t) = e^{kt}\overline{v}_1(t)$ and $v_2(t) = e^{kt}\overline{v}_2(t)$ functions defined above,

$$\varphi_1 = \frac{(2+\sigma_1)}{(\sigma_1+1)(\sigma_2+1)-1};$$

$$\varphi_2 = \frac{(2+\sigma_2)}{(\sigma_1+1)(\sigma_2+1)-1}$$

$$\xi = \frac{x}{[\tau_1(\tau)]^{1/2}}, \quad \tau_1(\tau) = \int_0^{\tau} \overline{v}_1^{\sigma_1}(y)dy.$$

Record $(a)_+$ means $(a)_+ = \max(0,a)$. These functions have the property of finite propagation speed [18, 21]. Therefore, in the numerical solution of task (2) when $\beta_1 > \sigma_1$ as initial approximation proposed functions $v_{i0}(t,x), i = 1,2$.

Created on input language Math Cad program allows you to visually trace the evolution process for different values of the parameters and data.

Numerical calculations show that in the case of arbitrary values $\sigma > 0, \beta > 0$ qualitative properties of solutions do not change. Listed below are the results of numerical experiments for different values of parameters (Figure 1 - 3).

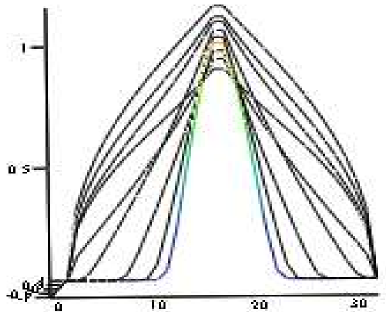
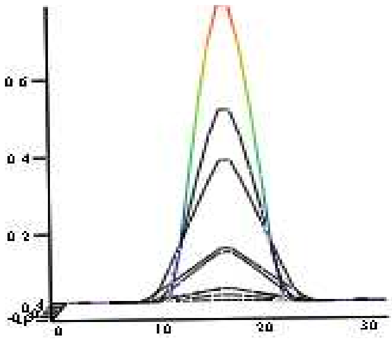
Values of Parameters	Results of Computational Experiment
$\sigma = 1.1$ $k_1 = 0.1$ $k_2 = 0.25$ $eps = 10^{-3}$	
$\sigma = 1.3$ $k_1 = 5$ $k_2 = 7$ $eps = 10^{-3}$	

Figure 1: Dynamics of Malthusian Population

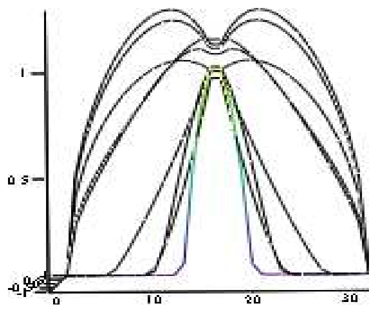
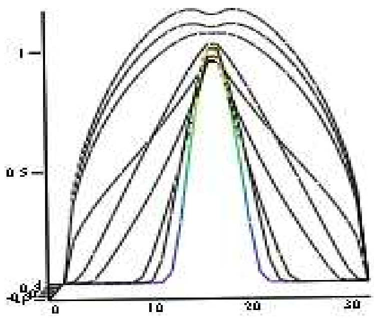
Values of Parameters	Results of Computational Experiment
$\sigma = 1.1$ $\beta_1 = 1, \quad k_1 = 7$ $\beta_2 = 1, \quad k_2 = 9$ $eps = 10^{-3}$	
$\sigma = 1.3$ $\beta_1 = 1, \quad k_1 = 3$ $\beta_2 = 1, \quad k_2 = 2$ $eps = 10^{-3}$	

Figure 2: Dynamics of the Logistic Population

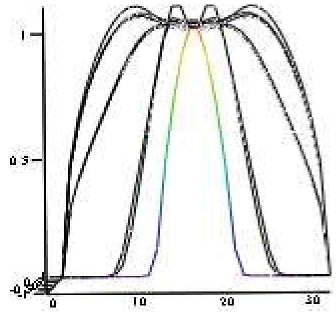
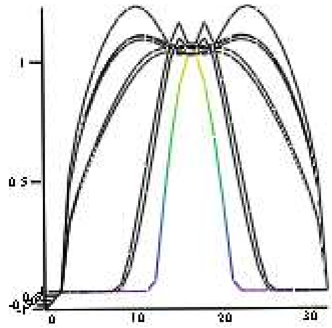
Values of Parameters	Results of Computational Experiment
$\sigma = 1.1$ $\beta_1 = 7, \;\; k_1 = 7$ $\beta_2 = 9, \;\; k_2 = 5$ $eps = 10^{-3}$	
$\sigma = 1.3$ $\beta_1 = 3, \;\; k_1 = 5$ $\beta_2 = 7, \;\; k_2 = 3$ $eps = 10^{-3}$	

Figure 3: Dynamics of Allee Effect

CONCLUSIONS

Thus, it can be expected that future theoretical and experimental research of excitable systems with cross-diffusion will make a significant contribution to the study of phenomena of self-organization in all of nonlinear systems from the micro and astrophysical systems to public social systems.

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